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AMPERE AND FARADAY: THE ELECTROMAGNETIC SYMMETRY

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List of symbols
(SI units are implied throughout)

B	magnetic flux density
D	electric flux density
E	electric field strength
e	electromotive force, e.m.f.
H	magnetic field strength
i	electric current
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	unit vectors
l	length
p	magnetic charge
q	electric charge
t	time
v_c, v_f	velocity (conductor, field)
v_1, v_2	velocity (forward, backward)
x, y, z	(suffixes) coordinate axes
Φ	magnetic flux
Ψ	electric flux

1 INTRODUCTION

The laws governing the basic electromagnetic processes are shown in Table 1 in the order in which they are usually presented to students. [In traditional treatments of 'flux-cutting e.m.f.', the conductor is assumed to be moving (across and) relative to the magnetic field, and in vector notation, the induced electric field is given by $\boldsymbol{E} = v_c \times \boldsymbol{B}$, where v_c is the conductor velocity. But in Table 1, equation (2) $\boldsymbol{E} = -v_f \times \boldsymbol{B}$ uses the field velocity v_f for the following reasons. In equation (3), it is the field that undergoes change, and in equation (5) of Table 2, to be derived in Section 2, it will be magnetic poles or 'charges' that travel through (and relative to) a coil. Hence the use of the field velocity in equation (2) ensures physical and mathematical uniformity with the other two equations.]

When equation (4) (Maxwell's law of 'magneto-electric induction'?) has been presented, attention is drawn to its striking similarity to equation (3), (Faraday's law of electromagnetic induction)[1]. The mutual reciprocality or symmetry between the two is obvious.

TABLE 1 *Electromagnetic laws.*

No.	Ampere's Law	Faraday's Law	Rate of
1	$\int H\,dl = i = \frac{dq}{dt}$ (1)		charge motion
2		$\boldsymbol{E} = -\boldsymbol{v}_f \times \boldsymbol{B}$ (2)	flux cut
3	$\int H\,dl = \frac{d\Psi}{dt}$ (4)	$\int E\,dl = -\frac{d\Phi}{dt}$ (3)	flux change

This raises the question of the possibility of the other two processes, described by equations (1) and (2), also having electromagnetic reciprocals. The questions are:

(a) Just as *electric charge-motion current* produces a *magnetomotive force (m.m.f.)* around it, equation (1) of Table 1, is there a '*magnetic charge-motion current*' inducing an *electromotive force (e.m.f.)* around it?

(b) Corresponding to *e.m.f.* induced by *magnetic flux cutting*, equation (2) of Table 1, is there an *m.m.f.* induced by *electric flux cutting*? (For example, if a straight bar of iron has relative movement across an electric field, will it have an m.m.f. induced in it?)

The following Sections seek to answer these questions.

2 MAGNETIC CURRENT INDUCING E.M.F.

2.1 *Magnetic charge and current*

In electromagnetic field theory, distinction is made between two broad classes of electric current: charge-motion current, which may be either by conduction or by convection, and displacement current, which may be in free space or in a material medium. Let us consider the possibility of magnetic charge-motion current.

An electric charge radiates electric flux. Conversely, a source (or destination) of electric flux is an electric charge. Can this idea be extended to the magnetic case? While a magnetic pole, say a north pole, is a source of magnetic flux across its exposed surface, it is also the destination of the same flux, via the associated south pole, across the unexposed surface bisecting the two poles. But to a conductor lying across the magnetic field, or a coil linking flux, the pole is only a source of flux and, as such, may be regarded as magnetic charge.

There are, of course, fundamental differences between electric charge and magnetic charge as defined above. Not only can electric charges of either polarity exist in isolation but, as elementary particles of nature, they form micro-constituents of matter. These properties enable them to be driven around by an electromotive force acting within an electric circuit composed of material media, and the process is the familiar conduction current.

On the other hand, magnetic charges not only always occur in pairs of opposite polarities but, as defined above, far from being elementary particles constituting matter, are themselves solid bodies formed by aggregate matter. These two facts make it impossible for magnetic charges to be driven around by a magnetomotive force acting in a magnetic circuit, ruling out the possibility of 'magnetic conduction current'. But is convection current possible?

[Electrically, materials are broadly of two kinds: those with free charges (conductors) and those without (insulators). An electrical potential difference V acting across a medium can have either of two effects depending upon the nature of the medium: in one that has free charges, it produces a current I, and in a medium with no free charges, it sets up a flux Ψ. Magnetically, however, all materials are basically of the same kind: none has free magnetic charges. Consequently, an m.m.f. acting in any material can only set up a flux Φ, never a 'magnetic current'.

Therefore an electric circuit, in general, can have both current-carrying parts and (electric) flux-carrying parts, while a magnetic circuit can only have flux-carrying parts. But the electric analogue of a magnetic circuit is an electric circuit composed entirely of current-carrying parts, i.e. a purely resistive circuit, because of the closed nature of the electric current path in such a circuit analogous to the closed magnetic flux lines.

These are perhaps the first and the most obvious aspects of electromagnetic '*asymmetry*' a learner encounters.]

2.2 *Magnetic convection current inducing e.m.f.*

Let us first clarify the concept of magnetic charge. The conventional concept is that of electric charge which, as a point source, radiates electric flux symmetrically in all directions. A magnetic pole not only is not a point source, but also is inseparably coupled by solid matter to its counterpart of opposite polarity.

Consider Fig. 1(a), which is the developed ('cylindrical' configuration converted to 'Cartesian') view of part of an alternator rotor showing one of the poles; an armature conductor is extended to form a loop threaded through an axial hole in the rotor shaft. In spite of its differences from an electric charge, as far as the loop is concerned, the pole represents a source of flux, at least over the active conductor part. The series of poles then form a string of solid magnetic charges of alternate polarities. As the rotor revolves, the movement of the string of charges constitutes an alternating magnetic convection current through the loop. Now,

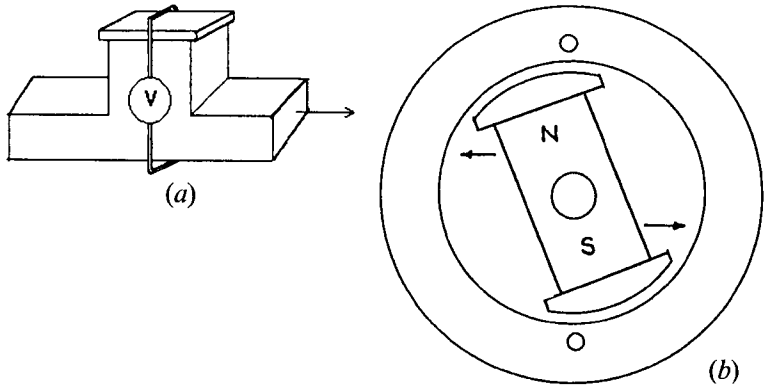

FIG. 1 Magnetic current through coil.

e.m.f. induced in loop = e.m.f. induced in conductor
= rate of flux cut by conductor
= rate of radial flux crossing loop
= rate of magnetic charge motion through loop
= magnetic convection current through loop

It is obvious that if the magnetic field were to extend beyond the active conductor, to part or the whole of the rest of the loop, the above result would still apply. (This will be proved presently, with reference to Fig. 2.) When the rotor completes one revolution, the loop will have swept over or covered a toroidal surface enclosing a volume. The total flux out of this volume, which is the total flux cut by the loop or time integral of the magnetic current through it (in volt-seconds or webers) is zero. But during the revolution, the magnetic current will have undergone as many cycles as the number of pole pairs.

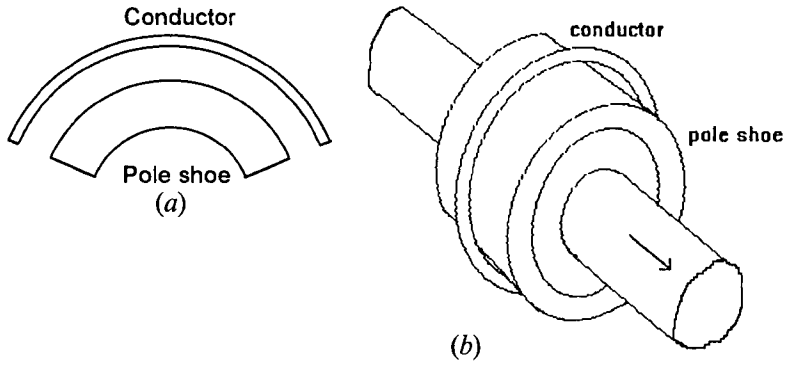

FIG. 2 Conversion of pole and conductor of Fig. 1(a) to cylindrical configuration.

Fig. 1(b) shows a 2-pole alternator with a full-pitched turn. Here magnetic charges of opposite polarities travel in opposite directions through the turn, so that the total rate of charge motion or magnetic convection current, and hence the induced e.m.f., are twice those in Fig. 1(a).

Now consider a more realistic situation: let the armature coil of the 2-pole alternator have a high degree of (short) chording or let it be a multipolar machine in which the armature coil spans only a small fraction of the periphery. In either case, the poles do not actually go through the coils, and the idea of magnetic current through coil apparently fails. But with reference to Fig. 1(a), consider each conductor of the coil separately. Let the developed rotor and the conductor be 'rolled' again, but now around an axis along the machine periphery, Fig. 2(a). At the end of this geometrical process, which is conversion from Cartesian back to cylindrical configuration, but now on a different axis, Fig. 2(b), the conductor becomes a circular loop lying in a radial plane. The poles become part of a cylindrical bar magnet, but with radial (rather than axial) magnetisation. Also, the magnetisation is of alternate polarities sinusoidally distributed along the length of the bar.

As the rotor rotates, the corresponding effect in the rolled diagram of Fig. 2(b) would be an axial travel of the cylindrical charges through the loop, constituting a magnetic convection current. The rate of charge motion can be once again shown to be the same as the rate of flux cut by the conductor in Fig. 1(a). We may therefore state that

$$\int E \, dl = e = -\frac{dp}{dt} \tag{5}$$

In words, the line integral of E around any closed path is equal to the negative of the *magnetic current* linked by that circuit. The direction of the e.m.f. in the loop implied in equation (5) obtained by Lenz's law is the same as that in the straight conductor obtained by Fleming's right hand rule. Equation (5) is thus the exact converse of Ampere's law for the magnetic effect of charge-motion electric current, equation (1). Accordingly, it is placed in the appropriate box in Table 2.

Note that here we are invoking neither the artifice of a fictitious unipolar magnetic charge (or 'monopole') in isolation nor the less fictitious, but microscopic, magnetic dipoles. We only need to broaden our concept of charge as a source of flux in space for all but closed surfaces. Particularly for a wire loop, magnetic charge is simply as real as magnetic flux. Once this is accepted, charge movement constituting current is logical.

A few points relating to the concept of magnetic current are worth noting. The first is that magnetic current, being a motion of solid and hence massive bodies, has to be mechanically (rather than 'magnetomotively') produced by an external agent of force (the prime mover), which makes it a means of electromechanical energy conversion. The second point is that it is always alternating and is a unidirectional motion of charges of alternate polarities, unlike alternating electric current which is flow(s) of unipolar charges in alter-

TABLE 2 *Complete set of electromagnetic laws.*

No.	Ampere's Law	Faraday's Law	Rate of
1	$\int H\,dl = i = \frac{dq}{dt}$ (1)	$\int E\,dl = e = -\frac{dp}{dt}$ (5)	charge motion
2	$\boldsymbol{H} = v_f \times \boldsymbol{D}$ (6)	$\boldsymbol{E} = -v_f \times \boldsymbol{B}$ (2)	flux cut
3	$\int H\,dl = \frac{d\Psi}{dt}$ (4)	$\int E\,dl = -\frac{d\Phi}{dt}$ (3)	flux change

nate directions. The third and final point is that, logically, $d\Phi/dt$ now becomes *magnetic displacement current.*

3 FLUX CUTTING M.M.F.

Fig. 3(a) shows the classic Ampere's law situation, an iron ring enclosing a current-carrying conductor. Let us convert this to Cartesian configuration, i.e. develop the iron ring into a straight bar, which would convert the conductor current into a current sheet of width equal to the bar length, Fig. 3(b). Can the m.m.f. in the bar be attributed to *electric flux cutting*? Apparently not, because there is no flux to be cut.

Let us leave this situation for the moment and consider instead its convection

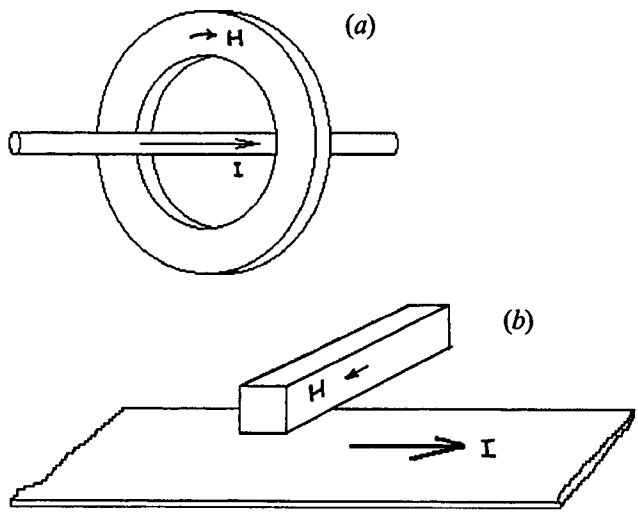

FIG. 3 Ampere's law: conduction current. (a) Cylindrical configuration; (b) Cartesian configuration.

counterpart, Fig. 4. Fig. 4(a) shows a uniformly charged wire moving longitudinally, constituting a convection current which will set up a magnetic field in the iron ring around it[2]. [At least one of the two, the charged wire or the magnetic ring, would have to be non-conducting; if both were conducting, their proximity would distort the charge distribution on each element of the wire as it approached and left the ring. But this requirement does not affect our argument.] Alternatively, the convection current may be an electron stream such as that in a cathode-ray tube. Fig. 4(b) is the developed or Cartesian configuration of Fig. 4(a): a bar of magnetic material lies parallel to an electrically charged sheet travelling across the bar. (The electric flux out of this charged sheet would all be upwards, with no flux downwards, because in the wire, there is no flux radially inwards.)

With a right combination of charge distribution and velocity in Fig. 4, the m.m.f. in the ring and the bar would be identical to those in the ring and the bar of Fig. 3. Now, there is obvious flux-cutting taking place in the convection current of Fig. 4, but there is none in the case of the conduction current of Fig. 3. Should not identical effects have identical causes?

This apparent paradox can be resolved as follows. Moving electric charges have *two* effects: not only a stationary magnetic field but also, of course, their own *travelling electric field.* (This is much easier to visualise in the magnetic case: moving magnetic charges produce not only a stationary electric field but also a *travelling magnetic field*; e.g. any of the revolving poles in an alternator will, if offered, carry an iron plate with it while inducing e.m.f. in the stationary armature conductors.)

In metallic conduction, which is unipolar, the moving electrons set up a

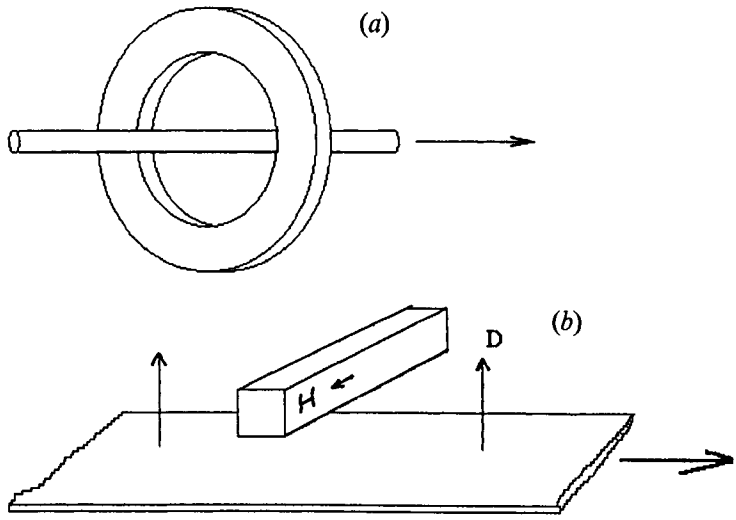

FIG. 4 Ampere's law: convection current. (a) Cylindrical configuration; (b) Cartesian configuration.

stationary magnetic field, but their travelling electric field is neutralised by the stationary electric field of the positive charges at rest. In semiconductors, liquids and gases, in which conduction is bipolar, charges of opposite polarities moving in opposite directions (with different velocities in general), both set up magnetic fields in the same direction, but again their electric fields neutralise each other.

In both types of conduction, therefore, the magnetic effect of electric flux cutting is evident, but the electric flux itself is not. This explanation is mathematically proved in the Section 7 (Appendix) by applying, *mutatis mutandis*, the theory of Dwight–Bewley paradox[3], which is the electromagnetic converse of the situation of Fig. 3(b). It will be clear that not only does electric flux cutting induce m.m.f., but the basic Ampere's law situation itself involves the process. We may therefore write, by analogy with equation (2),

$$\boldsymbol{H} = v_f \times \boldsymbol{D} \tag{6}$$

This equation fills the appropriate slot in Table 2 which is now complete and symmetrical.

4 EARLIER IDEAS ON ELECTROMAGNETIC SYMMETRY

Some writers (e.g. Ref. [1]) point out the similarity between equations (1) and (3). Such similarity obviously now exists between equations (1) and (5) on the one hand and equations (3) and (4) on the other. But in spite of their mathematical similarity, there are important differences between the two sets of equations. The first is that the charge motion currents of line 1 represent *rates of physical flow of material particles or bodies*, whereas the displacement currents of line 3 are *rates of change of non-material quantities.*

The other difference is less obvious but more significant. The equations of line 1, which apply to cylindrical 'charge-through' configurations, have their Cartesian flux-cutting counterparts in line 2; this implies that the magnetic circuit in which m.m.f. is induced and the electric circuit in which e.m.f. is induced by the processes of line 1 need not be closed. But the equations of line 3, which obviously apply to cylindrical configurations, do not have Cartesian equivalents because they are only applicable to closed circuits.

Another typical statement that purports to establish a somewhat different form of reciprocality on a physical basis is: '... a current-carrying conductor produces a magnetic field ... the reverse effect is also possible. That is, a magnetic field can produce a current in a closed circuit, but with the *important qualification* (italics present author's) that the magnetic field linking the circuit must be changing'[2], implying that equations (1) and (3) are actually the converse of each other.

But the 'important qualification' in this statement, together with the fact that what a changing magnetic field produces is an e.m.f. rather than a current, renders the intended 'converseness' inexact if not invalid. This is natural because, the exact converse of *electric current* producing *magnetic field*, equation (1), is *magnetic current* producing *electric field*, equation (5), rather than simply *magnetic field* producing *electric current.*

5 GENERAL

Since the directions involved in equation (1) [Figs. 3(a) and 4(a)] are related by the right-hand screw or grip rule, those in equation (5) [Figs. 1 and 2(b)] are related by a left-hand screw or grip rule. Similarly, since the directions in equation (2) are related by Fleming's right-hand rule, those in equation (6) [Figs. 3(b) and 4(b)] are related by a left-hand counterpart of Fleming's rule.

Equation (2) for flux-cutting e.m.f. can be obtained from Lorentz formula for force on a current-carrying conductor situated across a magnetic field[2]. No comparable origin or procedure seems to exist for arriving at equation (6) for flux-cutting m.m.f.

The reasons why equations (1) and (2), rather than equations (5) and (6), were formulated by the founders of electromagnetism, and why the need for the latter two equations never arose in theory or practice should form a good discussion topic for the classroom.

In traditional treatments, when the time comes for formulating equation (4), the concept of electric displacement current takes a great deal of explaining and, apparently, there are more than one way of arriving at it. But now, if equations (5) and (6) are covered before equation (3), students will automatically write in equation (4), without prompting, to fill up the last slot in Table 2. The meaning of $\partial\Psi/\partial t$ will then become much easier to impart, by analogy with $\partial\Phi/\partial t$.

But, of course, symmetry itself stops here. The very next step, a move on to the complete Maxwell's equations, brings us up against asymmetry. Faraday's law is

$$\int E\,\mathrm{d}l = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \qquad \text{[Equation (3) repeated]} \tag{7}$$

but Ampere–Maxwell law is

$$\int H\,\mathrm{d}l = I + \frac{\mathrm{d}\Psi}{\mathrm{d}t} \tag{8}$$

While there is no place for a magnetic charge-motion current in equation (7), the complete equation (8) includes electric charge-motion current I. As it is, students have to be convinced as to why, in equation (8), $\mathrm{d}\Psi/\mathrm{d}t$ should be added to I. But in the suggested treatment highlighting symmetry, the question to be answered might well be why I [whose effect, equation (1), can be found from the differential or point-function form of equation (4) itself, as shown in the Appendix] should be added to $\mathrm{d}\Psi/\mathrm{d}t$, thus upsetting symmetry!

6 CONCLUSIONS

The picture of basic electromagnetic processes currently presented in text books, Table 1, is incomplete. The idea propounded here, viz. that, in both electric and magnetic cases, the process of charge flow through a closed circuit and the process of flux cut by straight material are one and the same, fills the gaps in the picture, making it complete and fully symmetrical as follows:

An electric current produces an m.m.f. around it.
A magnetic current induces an e.m.f. around it.
A conductor cutting magnetic flux has e.m.f. induced in it.
An iron bar cutting electric flux has m.m.f. induced in it.

('Conductor' and 'iron bar' are intended for ease of visualisation.)

The full picture, presented in Table 2, should not only help students in the comprehension and retention of the concepts, but also enable them to appreciate the beauty of electromagnetic symmetry.

7 APPENDIX

It will now be shown that electric flux cutting occurs when a magnetic circuit encloses a conduction current as in Fig. 3. Consider the more general bipolar conduction first. In Fig. 3(b), let the electric flux density distributions due to the two oppositely moving charge distributions on the sheet be expressed as travelling waves

$$D_y = D_{y1} + D_{y2} = f_1(x - v_1 t) + f_2(x + v_2 t) \tag{9}$$

Here, the flux density space distributions as well as their velocities may be different, for purposes of generality, as indicated in Fig. 5. [Note that $f_1(x - v_1 t)$ represents the component moving in the positive x direction, i.e. forward moving field, and $f_2(x + v_2 t)$ the backward moving field.] In vector form,

$$\boldsymbol{D} = i D_x + j D_y + k D_z \tag{10}$$

with $D_x = D_z = 0$.

Since we are treating an electric current as a travelling electric field, the equation to be used is Maxwell's equation (4), rather than Ampere's, equation (1). The differential form of this equation is

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \tag{11}$$

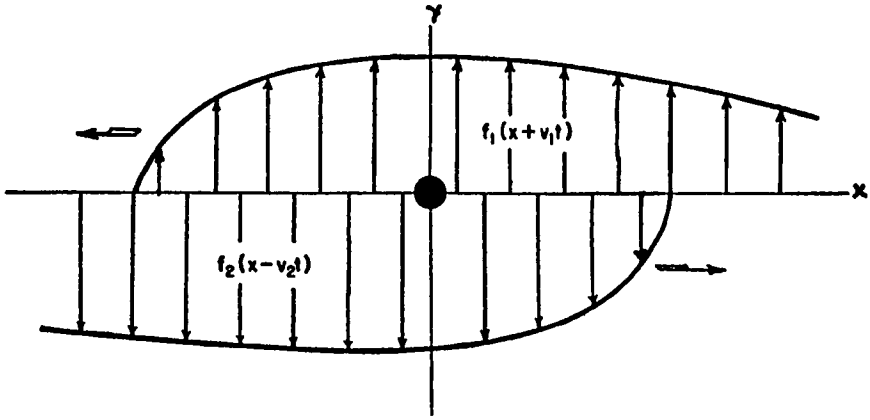

FIG. 5 Travelling waves of electric flux density.

which, when expanded, yields the non-zero component

$$\frac{\partial H_z}{\partial x} = -\frac{\partial D_y}{\partial t} = v_1 f_1'(x - v_1 t) - v_2 f_2'(x + v_2 t) \tag{12}$$

The solution to this differential equation, compatible with the implied boundary conditions, is

$$H_z = v_1 f_1(x - v_1 t) - v_2 f_2(x + v_2 t) \tag{13}$$

Now, in real situations of conduction current, the two components of flux density are equal and opposite and constant at all points and at all times, so that

$$D_y = 0 \tag{14}$$

which, when imposed on equation (9) yields

$$f_1(x - v_1 t) = -f_2(x + v_2 t) = D \quad \text{(say)} \tag{15}$$

Using equation (15) in equation (13),

$$H_z = v_1 D + v_2 D = (v_1 + v_2) D \tag{16}$$

M.m.f. in the bar is

$$H_z l = (v_1 + v_2) D l$$

= total rate of electric flux cut by the bar

= total current in the sheet

Transferred to the cylindrical configuration of Fig. 3(a), this is the conductor current and the m.m.f. in the ring. Hence

m.m.f. in bar due to flux cutting

= m.m.f. in ring due to enclosed current. (17)

This is the most general result.

Next let the conductor current in Fig. 3 be unipolar, such as in metals. Again there is no resultant electric field, so that equations (14) and (15) are still true. But the positive charge distribution is now stationary so that $v_1 = 0$ in equation (16). This yields

$$H_z = v_2 D \tag{18}$$

Applying the logic following equation (16) leads to equation (17) again.

That there is flux cutting in the case of the (unipolar) convection current of Fig. 4 is obvious. All the same, for completeness, let us apply the above theory to this case. There will be electric field of only one (say positive) polarity, D_{y1}, which will be evident since $D_{y2} = 0$. Equations (9) and (15) become

$$D_y = D_{y1} = f_1(x - v_1 t) = D \tag{19}$$

which, when used in equation (13) gives

$$H_z = v_1 f_1(x - v_1 t) = v_1 D \tag{20}$$

This leads to equation (17) again.

The magnitudes and directions of the m.m.f.s in all three cases conform to Ampere's law and the right-hand screw/grip rule. Equation (6) is the general vector form of equations (16), (18) and (20).

8 ACKNOWLEDGEMENT

The author learnt the essentials of electromagnetism from the late Prof. G. W. Carter of Leeds University and from his classic book *The Electromagnetic Field in its Engineering Aspects*[1].

9 REFERENCES

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ABSTRACTS – ENGLISH, FRENCH, GERMAN, SPANISH

Ampere and Faraday: the electromagnetic symmetry

Emf induced by flux-cutting is interpreted as that in a closed circuit due to an enclosed 'magnetic (convection) current'. Conversely, the m.m.f. in a closed magnetic circuit enclosing a current is interpreted as due to cutting of electric flux. These interpretations make the picture of electromagnetic phenomena complete and symmetrical.

Ampère et Faraday: la symétrie électromagnétique

La f.e.m. induite par un déplacement dans un flux est interprétée comme celle créée dans un circuit fermé due à un "courant magnétique (de convection)". Inversément la f.m.m. dans un circuit magnétique fermé entourant un courant est interprétée comme due à un déplacement dans un flux électrique. Ces interpétations rendent une image complète et symétrique des phénomènes électromagnétiques.

Ampere und Faraday: Die elektromagnetische Symmetrie

Elektromotorische Kraft, die durch Flußschnitt induziert wurde, wird als die in einem geschlossenen Schaltkreis interpretiert, aufgrund eines geschlossenen 'magnetischen (Konvektions) Strom'. Umgekehrt wird die magnetomotorische Kraft in einem geschlossenen magnetischen Schaltkreis, die einen Strom einschließt, interpretiert als durch Schneiden des elektrischen Flusses verursacht. Diese Interpretationen vervollständigen das Bild elektromagnetischer Phänomene und machen sie symmetrisch.

Ampere y Faraday: La simetría electromagnética

La fuerza electromotriz inducida por corte de corriente en un circuito cerrado es interpretada como la debida a una "corriente magnética" encerrada (covección). Recíprocamente, la fuerza magnetomotriz en un circuito magnético cerrado se interpreta como debida a un cese del flujo eléctrico encerrado. Estas interpretaciones hacen que el cuadro de los fenómenos electromagnéticos sea completo y simétrico.